Discrete Mathematical Structures:

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DMS Notes

Module-I

Fundamentals of Logic:

Propositions:

Proposition is a statement which, in a given context, can be said to be either true or false, but not both.

Eg: Bengaluru is in Karnataka ✓ True

3 is a prime number ⇒ True

Seven is divisible by 3 ⇒ False

Every rectangle is a square ⇒ False

Note: Not all sentences are propositions.

Eg: 1. Take a triangle ABC ⇒ Not a statement ⇒ Not proposition.

2. $xy = yx$ ⇒ statement, but we can't say this statement is true or false unless we know the values.

↳ Hence not a proposition.

⇒ propositions are usually represented by small letters as

⇒ p, q, r, s ...

* Truth value: The truth or falsity of a proposition.

→ If proposition is true, its truth value is 1 (T)

—"— false —"— is 0 (F)

i.e., p = 3 is a prime number

$\boxed{p=1}$ or $\boxed{p=T}$

q = Every square is a rectangle

$\boxed{q=0}$, $\boxed{q=F}$

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Logical connectives & truth tables:

* propositions used with words as not, and, if---then and if and only if, gives new propositions called compound propositions.
* words such as, not, and, if---then, & if and only if are called logical connectives.
* original propositions from which compound proposition is obtained are called components or primitives of compound proposition.
* propositions which do not contain any logical connective are called simple propositions.

Negation: proposition obtained by inserting the word 'not' at an appropriate place in a given proposition is called the negation of the given proposition.

Denoted by: $\neg p$. (not p)

Eg: proposition → "3 is a prime number", denoted by p.

i.e, p: 3 is a prime number

negation of a proposition

negation of 'p' is "3 is not a prime number"

i.e, $\neg p$: 3 is not a prime number

Truth table for Negation

p	$\neg p$
0	1
1	0

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Conjunction:

→ A compound proposition obtained by combining two given propositions by inserting the word 'and' in between them is called the conjunction of the given propositions.

Denoted by $p \wedge q$ (p and q).

* The conjunction $p \wedge q$ is true only when 'p' is true and 'q' is true; in all other cases it is false.

i.e, truth value of $p \wedge q$ is 1 only when truth value of 'p' is 1 & truth value of 'q' is 1. In all other cases truth value of $p \wedge q$ is '0'.

Truth table for conjunction:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Eg: p: $\sqrt{2}$ is an irrational number T
q: 9 is a prime number F
r: All triangles are equilateral F
s: 2+5 = 7 T

Here 'p' & 's' are true propositions
q & r are false propositions

then,

$p \wedge q$: $\sqrt{2}$ is an Irrational number (true) and 9 is a prime number (false)

$p \wedge q \Rightarrow$: false

$q \wedge r$: 9 is a prime number (false) and All triangles are equilateral (false)

$q \wedge r \Rightarrow$ false

$r \wedge s$: All triangles are equilateral (false) and 2+5 = 7 (true)

$r \wedge s$: false

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④

$s \wedge p$: $2+5=7$ and $\sqrt{2}$ is an Irrational number
(true) (true)

$s \wedge p$: true

Disjunction: A compound proposition obtained by combining two given propositions by inserting the word 'OR' between them is called the disjunction of the given propositions.

Denoted by $p \vee q$ (p or q).

Rule: The disjunction $p \vee q$ is false only when 'p' is false & 'q' is false, in all other cases it is true.

i.e., $p \vee q$: 0 when p is '0' & q is '0'.
all other cases, truth value of $p \vee q$ is 1.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Eg: p: $\sqrt{2}$ is an irrational number $\Rightarrow$ true
q: 9 is a prime number $\Rightarrow$ F
r: All triangles are equilateral $\Rightarrow$ F
s: $2+5=7$ $\Rightarrow$ T

then,

$p \vee q$: $\sqrt{2}$ is an Irrational number or 9 is a prime number.
$\Rightarrow$ true

$q \vee r$: q is a prime number or All triangles are equilateral
$\Rightarrow$ False

$r \vee s$: All triangles are equilateral or $2+5=7$
$\Rightarrow$ true

$s \vee p$: $2+5=7$ or $\sqrt{2}$ is an Irrational number
$\Rightarrow$ true

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Exclusive disjunction:

In the disjunction $p \vee q$ of two propositions p and q, the symbol 'v' is used in the Inclusive sense.

i.e, $p \vee q$ is taken to be true when p or q or both p and q are true.

But, sometimes it is require the use of word 'or' in Exclusive sense, i.e, the compound proposition "p or q" to be true only when either 'p' is true or 'q' is true, but not both.

Exclusive or is denoted by '$\underline{\vee}$'. or '$\overline{\vee}$'

Denoted by $p \underline{\vee} q$ (Either 'p' is true or 'q' is true buth not both).

Truth table:

p	q	$p \underline{\vee} q$
0	0	0
0	1	1
1	0	1
1	1	0

Eg:

p: $\sqrt{2}$ is an Irrational number

q: $2+3=5$

r: 9 is a prime number

s: All triangles are isoceles

then,

$p \underline{\vee} q$: Either $\sqrt{2}$ is true an Irrational number or $2+3=5$ but not both.

$q \underline{\vee} r$: Either $2+3=5$ or 9 is a prime number, but not both

$r \underline{\vee} s$: Either 9 is a prime number or all triangles are isoceles but not both.

$s \underline{\vee} p$: Either all triangles are isoceles or $\sqrt{2}$ is an Irrational number but not both.

since, p & q are true

r & s are false

$p \underline{\vee} q \Rightarrow$ False

$r \underline{\vee} s \rightarrow$ False

$q \underline{\vee} r \rightarrow$ True

$s \underline{\vee} p \rightarrow$ True

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Conditional:

A compound proposition obtained by combining two given propositions by using the words "if" and "then" at appropriate places is called a conditional.

Denoted by: $p \to q$ (If 'p' then 'q')

$q \to p$ (If 'q' then 'p')

Note: $p \to q$ & $q \to p$ are not same.

Rule: conditional $p \to q$ is false only when 'p' is true & 'q' is false in all other cases it is true.

i.e, $p \to q$ is '0' when truth value of 'p' is 1 & the truth value of 'q' is '0'. In all other cases, the truth value of $p \to q$ is 1.

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Note: $p \to q$ is taken to be true when

(i) p is false & q is true

(ii) p & q are both false

p: 2 is a prime number q: 3 is a prime number

r: 6 is a perfect square s: 9 is a multiple of 6.

Here, p & q $\Rightarrow$ true

r & s $\Rightarrow$ false

$p \to q$: If 2 (T) is a prime number then 3 is a prime number (T)

$\Rightarrow$ True

$p \to r$: If 2 is a prime number (T) then 6 is a perfect square (F)

$\Rightarrow$ false

$r \to p$: If 6 is a (F) perfect square then 2 is a prime no. (T)

$\Rightarrow$ True

$r \to s$ $\to$ If 6 is a perfect (F) square then 9 is a multiple of '6'. (F)

$\Rightarrow$ True

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Biconditional

Let 'p' and 'q' are two propositions. Then the conjunction of the conditional $p \to q$ and $q \to p$ is called the biconditional of p and q.

Denoted by: $p \leftrightarrow q$ which is same as,

$$(p \to q) \wedge (q \to p)$$

$p \leftrightarrow q$ (If 'p' then q and if q then p).

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

$p \leftrightarrow q$ is true only when both p & q have same truth values.

Eg: p: 2 is a prime number q: 3 is a prime number
r: 6 is a perfect square s: 9 is a multiple of 6.

$p \leftrightarrow q$: True , $p \leftrightarrow r$: False , $q \leftrightarrow r$: False
$r \leftrightarrow s$: True $p \leftrightarrow s$: False $q \leftrightarrow s$: False

combined truth table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \to q$	$p \leftrightarrow q$
0	0	1	0	1	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

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Examples:

(1) p: A circle is a conic q: $\sqrt{5}$ is a real number
r: Exponential series is convergent

Express the following compound propositions in words.

(i) $p \wedge (\neg q)$ (ii) $(\neg p) \vee q$ (iii) $p \veebar (\neg q)$

(iv) $q \rightarrow (\neg p)$ (v) $p \rightarrow (q \veebar r)$ (vi) $\neg p \leftrightarrow q$

Ans:

(i) $p \wedge (\neg q)$: A circle is a conic and $\sqrt{5}$ is not a real number

(ii) $(\neg p) \vee q$: A circle is not a conic or $\sqrt{5}$ is a real number

(iii) $p \veebar (\neg q)$: ~~A circle is a conic~~

Either a circle is a conic or $\sqrt{5}$ is not a real number (But not both)

(iv) $q \rightarrow (\neg p)$: If $\sqrt{5}$ is a real number then a circle is not a conic

(v) $p \rightarrow (q \veebar r)$: If a circle is a conic then either $\sqrt{5}$ is a real number or the exponential series is convergent (but not both).

(vi) $\neg p \leftrightarrow q$: ~~A circle~~

If a circle is not a conic then $\sqrt{5}$ is a real number, and if $\sqrt{5}$ is a real number then a circle is not a conic.

(2) construct the truth tables for the following compound propositions:

① $p \wedge (\neg q)$ ② $(\neg p) \vee q$, ③ $p \rightarrow (\neg q)$ ④ $(\neg p) \veebar (\neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$p \rightarrow (\neg q)$	$(\neg p) \veebar (\neg q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

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(9) (3)

Let p and q be primitive statements for which the conditional $p \to q$ is false. Determine the truth values of the following compound propositions.

(1) $p \wedge q$ (2) $(\neg p) \vee q$ (3) $q \to p$ (4) $(\neg q) \to (\neg p)$

given: $p \to q$ is false

i.e, if 'p' is true and 'q' is false then $p \to q$ is false

(5) $p \to 1$ (true)
$q \to 0$ (false)

(1) $p \wedge q \Rightarrow$ truth value of $p \wedge q$ is '0'
1 0

(2) $(\neg p) \vee q$ truth value → 0
0 0

(3) $q \to p$ truth value → 1
0 1

(4) $(\neg q) \to (\neg p)$ truth value → 0
1 0

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(4) Let p, q and r be propositions having truth values 0, 0 and 1. Find the truth values of the following compound propositions.

given: $p \nRightarrow 0$, p: 0, q: 0, r: 1

(1) $(p \vee q) \vee r$

$\begin{bmatrix} 0 & 0 & \vee r \\ 0 & & 1 \\ & 1 & \end{bmatrix}$ → rough work

Ans: since 'p' is false & q is false

$p \vee q$ (p or q) ⇒ False

Now, $p \vee q \Rightarrow$ false, & r is true

∴ $(p \vee q) \vee r \Rightarrow$ True ie, 1, truth value = 1

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(2) $(p \wedge q) \wedge r$

given: p is '0' (false)
q is '0' (false)
r is '1' (true)

$\therefore (p \wedge q) \Rightarrow$ false

$\therefore (p \wedge q) \wedge r$

$\Rightarrow$ false truth value = 0

(3) $(p \wedge q) \rightarrow r$

$p \rightarrow$ false, $q \Rightarrow$ false
(0) (0)

$\therefore (p \wedge q) \Rightarrow$ false

$(p \wedge q) \rightarrow r$

0 1

true truth value = 1

(4) $p \rightarrow (q \wedge r)$

since, $q \Rightarrow$ false
$r \Rightarrow$ true

$(q \wedge r) \Rightarrow$ false (0)

WKT, $p \rightarrow$ false

$\therefore p \rightarrow (q \wedge r)$

0 0 $\Rightarrow$ true

~~false~~ truth value = 01

(5) $p \wedge (r \rightarrow q)$

$r \rightarrow$ true, $q \rightarrow$ false

WKT, if $(r \rightarrow q) \Rightarrow$ false (0).
1 0

$\therefore p \wedge (r \rightarrow q)$

$\Rightarrow$ false truth value = 0

(6) $p \rightarrow (q \rightarrow (\neg r))$

$\neg r \Rightarrow$ false (0)

$q \rightarrow (\neg r) \Rightarrow 1$
0 0 (true)

$p \rightarrow$ false

$\therefore p \rightarrow (q \rightarrow (\neg r))$

$\Rightarrow$ true
value = 1

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(5) Find the possible truth values of p, q & r in the following cases:

(1) $p \to (q \vee r)$ is false (2) $p \wedge (q \to r)$ is true

Ans: $p \to (q \vee r)$ can be false only when 'p' is true and $(q \vee r)$ is false

& $q \vee r$ is false only when both are false.

∴ true value of p is 1
q is 0
r is 0

Ans: $p \wedge (q \to r)$ is true

$p \wedge (q \to r)$ is true only when 'p' is true and $q \to r$ is true.

Also, $q \to r$ is true when * q is false and r is false
* q is false & r is true
* q is true & r is true

Truth values are:

$p \leftrightarrow$ p is 1

p	q	R
1	1	1
1	0	1
1	0	0

(5) construct the truth tables for the following compound propositions:

(i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$

* There are 3 primitive propositions: p, q, r
Each of the 3 primitives has two truth values.

∴ we can have $2^3 = 8$ sets of possible truth values of p, q, r.

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p	q	r	p∨q	(p∨q)∧r	q∧r	p∨(q∧r)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

	4	2	1
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

(7) construct the truth tables for the following compound propositions:

(i) $(p \wedge q) \rightarrow (\neg r)$ (ii) $q \wedge ((\neg r) \rightarrow p)$

p	q	r	¬r	p∧q	(p∧q)→(¬r)	(¬r)→p	q∧((¬r)→p)
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

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(13) (8) If a proposition 'q' has the truth value 1, determine all truth value assignments for the primitive propositions p, r and s for which the truth value of the following compound proposition is 1.

$$[q \rightarrow \{(\neg p \vee r) \wedge \neg s\}] \wedge \{\neg s \leftrightarrow (\neg r \wedge q)\}$$

Given: $q = 1$

Truth value of the given compound proposition = 1

Consider, $u = q \rightarrow \{(\neg p \vee r) \wedge \neg s\}$

$v = \{\neg s \leftrightarrow (\neg r \wedge q)\}$

Rough work

$u \wedge v$

at $u \wedge v = 1$

* Since, the truth value of compound proposition is '1' & we have $u \wedge v$, hence the values of both 'u' and 'v' should be '1'

WKT, $q = 1$, also $u = 1$

$\therefore$ & $(\neg p \vee r) \wedge \neg s$, truth value must be '1'

at $(\neg p \vee r) \wedge \neg s$ is 1,

$(\neg p \vee r)$ is '1'

$\neg s$ is 1

$\therefore \boxed{s = 0}$

* Since, $\neg s$ has the value '1', & v has value '1' $\therefore (\neg r \wedge q)$ must be 1

& WKT, $q = 1$, $\therefore \neg r$ value must be '1'

$\therefore \boxed{r = 0}$

* Since, we have $r = 0$

& $(\neg p \vee r)$

$\therefore \neg p$ must be '1'

$\therefore \boxed{p = 0}$

Hence, $\boxed{p = 0,\ q = 1,\ r = 0,\ s = 0}$ //

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(14) (9) Indicate how many rows are needed in the truth table for the compound proposition.

$$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$$

Find the truth value of this proposition if 'p' and 'r' are true and q, s, t are false.

Soln: The given compound proposition contains five premitives p, q, r, s, t.

∴ the no. of possible combinations of the truth values of these components which we have to consider is $\boxed{2^5 = 32}$

∴ 32 rows are needed in the truth table for the given compound proposition.

given: $p \Rightarrow$ true (1) $\quad q, s, t \Rightarrow$ false (0).

$r \Rightarrow$ true (1)

Therefore, $\neg q = 1$ (true)

$\neg r = 0$ (false)

$$\therefore \underset{\Rightarrow \text{ true (1)}}{(p \vee \neg q)} \qquad \underset{\Rightarrow \text{ false (0)}}{(\neg r \wedge s)} \qquad \underset{\Rightarrow \text{true (1)}}{(\neg r \wedge s) \rightarrow t}$$

i.e,

since, $p \Rightarrow$ true and $\neg q \Rightarrow$ true

∴ $p \vee \neg q$ is true

since, $\neg r$ is false, s is false

$\neg r \wedge s$ is false.

and, $(\neg r \wedge s)$ is false & t is false

$(\neg r \wedge s) \rightarrow t$ is true

∴ as $p \vee \neg q$ is true

and $(\neg r \wedge s) \rightarrow t$ is true

By using biconditional,

compound statement,

$$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$$

is true i.e, truth value = 1

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Tautology; contradiction:

A compound proposition which is true for all possible truth values of its components is called a tautology.

A compound ~~statement~~ proposition which is false for all possible truth values of its components is called a contradiction or an absurdity.

A compound proposition that can be true or false is called a contingency. i.e, contingency is a compound proposition which is neither a tautology nor a contradiction.

(1) prove that, for any proposition 'p', the compound proposition $p \vee \neg p$ is a tautology and compound proposition $p \wedge \neg p$ is a contradiction.

Soln: construct the truth table for $p \vee \neg p$ & $p \wedge \neg p$.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

∴ By the observation, $p \vee \neg p$ is always true, hence it is a tautology.

And, $p \wedge \neg p$ is always false; hence it is a contradiction.

(2) Show that, for any two propositions p and q,

(i) $(p \vee q) \vee (p \leftrightarrow q)$ is a tautology.

(ii) $(p \vee q) \wedge (p \leftrightarrow q)$ is a contradiction

(iii) $(p \vee q) \wedge (p \rightarrow q)$ is a contingency.

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(i) $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology. (ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction.

Sol: construct the truth table

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \vee (p \leftrightarrow q)$	$(p \veebar q) \wedge (p \leftrightarrow q)$
0	0	0	1	1	0
0	1	1	0	1	0
1	0	1	0	1	0
1	1	0	1	1	0

From the above table, we can see that, for all possible truth values of p and q, the compound proposition $(p \veebar q) \vee (p \leftrightarrow q)$ is always 1 and truth value of $(p \veebar q) \wedge (p \leftrightarrow q)$ is always '0'.

Hence we can conclude that,

$(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology.

& $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction.

(iii) $(p \veebar q) \wedge (p \rightarrow q)$ is a contingency.

Sol:

p	q	$p \veebar q$	$p \rightarrow q$	$(p \veebar q) \wedge (p \rightarrow q)$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	0	1	0

From the above table, we find that

$(p \veebar q) \wedge (p \rightarrow q)$ can have truth value '1' or '0'.

∴ the given compound proposition is neither a tautology nor a contradiction; it is a contingency.

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(3) Show that, for any propositions p and q, the compound proposition $p \to (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$ is a contradiction.

Soln: prepare the truth tables.

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \to (p \vee q)$ ✓ tautology	$(\neg p \wedge q)$	$p \wedge (\neg p \wedge q)$ ✓ contradiction
0	0	1	0	0	1	0	0
0	1	1	1	0	1	1	0
1	0	0	1	0	1	0	0
1	1	0	1	1	1	0	0

From the above table, note that, for all possible truth values 'p' and 'q', the compound proposition $p \to (p \vee q)$ is true.
And the compound proposition $p \wedge (\neg p \wedge q)$ is false.
Therefore, $p \to (p \vee q)$ is a tautology.
$p \wedge (\neg p \wedge q)$ is a contradiction.

(4) Show that the truth values of the following compound propositions are ~~idep~~ Independent of the truth values of these components:

(i) $\{p \wedge (p \to q)\} \to q$ (ii) $(p \to q) \leftrightarrow (\neg p \vee q)$

Soln: First prepare the truth table.

p	q	$p \to q$	$\neg p \wedge (p \to q)$	$\{p \wedge (p \to q)\} \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

→ Tautology.

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(ii)

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

↳ Tautology

From above two tables, the truth value of the given compound propositions is 1.

* We may observe that the truth values for the components of both compound propositions are different but the truth values of compound propositions are is 1, hence the compound propositions are Independent of the truth values of their components.

* Furthermore both the compound propositions are tautologies.

(5) prove that for any propositions 'p' and 'q' the compound proposition $[(\neg q) \wedge (p \to q)] \to (\neg p)$ is a tautology.

Soln: prepare the truth table.

p	q	$\neg p$	$\neg q$	$(p \to q)$	$[(\neg q) \wedge (p \to q)]$	$((\neg q) \wedge [p \to q]) \to (\neg p)$
0	0	1	1	1	1	1
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	1	0	0	1	0	1

* By observing truth table the last column gives all the truth value, hence given proposition is tautology. //

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(6) prove that for any propositions p, q, r the compound proposition $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

(7) prove that for any propositions p, q, r the compound proposition $\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$ is a tautology.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

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20 (8) prove that, for any propositions p, q, r, the compound proposition $[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$ is a tautology.

Sol: prepare the truth table.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}$	$(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\} \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

given proposition is a tautology.

Logical Equivalence

Two propositions u and v are said to be logically equivalent whenever u and v have the same truth value.

Or

The biconditional $u \leftrightarrow v$ is a tautology.

Represe Logical equivalence is represented as $u \Leftrightarrow v$.

Symbol '$\Leftrightarrow$' stands for "logically equivalent to".

* When the proposition u and v are not logically equivalent, then it is represented as $u \not\Leftrightarrow v$.

Note: Logically equivalent propositions are treated as Identical propositions.

(1) Let 'x' be a specified positive Integer. consider the following propositions:

p: x is an odd Integer q: x is not divisible by 2.

are p and q logically equivalent?

Ans: Note that p and q have the same truth values, hence p and q are logically equivalent.

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(21) (2) For any two propositions p, q, prove that $(p \to q) \Leftrightarrow (\neg p) \vee q$

Soln: prepare truth table.

p	q	$p \to q$	$\neg p$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

The above truth table reveals that $p \to q$ and $(\neg p) \vee q$ have the same truth values for all possible truth values of p & q.

$\therefore (p \to q) \Leftrightarrow (\neg p) \vee q$.

(3) Prove that, for any propositions p and q, the compound propositions $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent.

→ prepare the truth table.

p	q	$p \vee q$	$p \veebar q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	0	0	1	1	1	0
0	1	1	1	1	0	1	1
1	0	1	1	0	1	1	1
1	1	1	0	0	0	0	0

From the above truth table, the columns 4 and 8 are having same truth values which indicates $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent.

i.e, $(p \veebar q) \Leftrightarrow \{(p \vee q) \wedge (\neg p \vee \neg q)\}$

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(4) prove that, for any 3 propositions p, q, r,
$[p \to (q \wedge r)] \Leftrightarrow [(p \to q) \wedge (p \to r)]$

Prepare the truth table.

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1

In the above table columns 5 and 8 proves that $[p \to (q \wedge r)]$ and $[(p \to q) \wedge (p \to r)]$ are logically equivalent.

(5) prove that, for any three propositions p, q, r.
$[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$

p	q	r	$(p \vee q)$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

In the above table columns 5 and 8 proves that $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are having same truth values.

Hence,

$$(p \vee q) \to r \Leftrightarrow [(p \to r) \wedge (q \to r)]$$

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(6) Examine whether the compound proposition $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ is a tautology.

→ prepare the truth table.

p	q	r	$(p \vee q)$	$(p \vee q) \rightarrow r$	$\neg r$	$\neg(p \vee q)$	$\neg r \rightarrow \neg(p \vee q)$
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

In the above table $(p \vee q) \rightarrow r$ and $\neg r \rightarrow \neg(p \vee q)$ have the same truth values in all possible conditions. ∴ propositions are logically equivalent.

Hence, $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ is not a tautology.

(7) Show that the compound propositions $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

→ prepare the truth table.

p	q	r	$\neg q$	$\neg r$	$\neg q \vee r$	$q \wedge \neg r$	$p \wedge (\neg q \vee r)$	$p \vee (q \wedge \neg r)$
0	0	0	1	1	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	0	1	0	1
0	1	1	0	0	1	0	0	0
1	0	0	1	1	1	0	1	1
1	0	1	1	0	1	0	0	1
1	1	0	0	1	1	1	1	1
1	1	1	0	0	1	0	0	1

In the above truth table last two columns don't have the same truth values, hence both the propositions are not logically equivalent.

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The laws of logic

T_0 denotes tautology
F_0 denotes contradiction

1. Law of double negation
For any proposition P.
(a) $(\neg\neg P) \Leftrightarrow P$.

2. Idempotent laws:
For any proposition P.
(a) $(P \vee P) \Leftrightarrow P$, (b) $(P \wedge P) \Leftrightarrow P$

3. Identity laws:
For any proposition P.
(a) $(P \vee F_0) \Leftrightarrow P$, (b) $(P \wedge T_0) \Leftrightarrow P$

4. Inverse laws:
For any proposition P.
(a) $(P \vee \neg P) \Leftrightarrow T_0$ (b) $(P \wedge \neg P) \Leftrightarrow F_0$

5. Domination laws:
For any proposition P.
(a) $(P \vee T_0) \Leftrightarrow T_0$, (b) $(P \wedge F_0) \Leftrightarrow F_0$

6. Commutative laws:
For any two propositions p and q
(a) $(p \vee q) \Leftrightarrow (q \vee p)$ (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

7. Absorption laws:
For any propositions p and q
(a) $[P \vee (P \wedge q)] \Leftrightarrow P$ (b) $[P \wedge (P \vee q)] \Leftrightarrow P$

8. Demorgan laws:
(a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
(b) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

9. Associative laws:
For any 3 propositions
p, q, r
(a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

10. Distributive laws:
For any propositions p, q, r,
(a) $P \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

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(25) The law of double negation, Idempotent and commutative laws are trivially true.
other laws can be verified with the aid of the truth tables.

Identity law:

WKT, $T_0 \Rightarrow$ Tautology, $F_0 \Rightarrow$ contradiction

* For the tautology all the possible conditions are true.
* For the contradiction all the possible conditions are false.

Hence,

Truth table:

(a)

P	F_0	$P \vee F_0$	$\Rightarrow P$
1	0	1	
0	0	0	

i.e, $(P \vee F_0) \Leftrightarrow P$
Hence the proof

(b) $(P \wedge T_0) \Leftrightarrow P$

P	T_0	$P \wedge T_0$	$\Rightarrow P$
1	1	1	
0	1	0	

$(P \wedge T_0) \Leftrightarrow P$ Hence the proof

Inverse law:

(a) $(P \vee \neg P) \Leftrightarrow T_0$

P	$\neg P$	$P \vee \neg P$
1	0	1
0	1	1

All the values are true
hence $(P \vee \neg P)$ is equivalent to T_0 i.e, $(P \vee \neg P) \Leftrightarrow T_0$

(b) $(P \wedge \neg P) \Leftrightarrow F_0$

P	$\neg P$	$P \wedge \neg P$
1	0	0
0	1	0

All the values are false
hence $(P \wedge \neg P)$ is equivalent to F_0, i.e,
$(P \wedge \neg P) \Leftrightarrow F_0$

Domination laws:

(a) $(P \vee T_0) \Leftrightarrow T_0$

P	T_0	$P \vee T_0$
1	1	1
0	1	1

Here T_0 is dominated over P //

(b) $(P \wedge F_0) \Leftrightarrow F_0$

P	F_0	$P \wedge F_0$
1	0	0
0	0	0

Here F_0 is dominated over P //

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Commutative Laws:

(a) $(p \vee q) \Leftrightarrow (q \vee p)$

p	q	p∨q	q∨p
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Here columns 3 & 4 are same hence $(p \vee q) \Leftrightarrow (q \vee p)$

(b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

p	q	p∧q	q∧p
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Here columns 3 and 4 are same hence, $(p \wedge q) \Leftrightarrow (q \wedge p)$

Absorption laws:

(a) $[p \vee (p \wedge q)] \Leftrightarrow p$

p	q	p∧q	p∨(p∧q)
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Hence the proof

(b) $[p \wedge (p \vee q)] \Leftrightarrow p$

p	q	p∨q	p∧(p∨q)
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

Hence the proof

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27) Demorgan laws

(a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

Same

Hence the proof

(b) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Same

Hence the proof

Associative laws:

(a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$(p \vee q)$	$(p \vee q) \vee r$	
0	0	0	0	0	0	0	
0	0	1	1	1	0	1	This proof proves, $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
0	1	0	1	1	1	1	
0	1	1	1	1	1	1	
1	0	0	0	1	1	1	
1	0	1	1	1	1	1	
1	1	0	1	1	1	1	
1	1	1	1	1	1	1	

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(28)

(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$	$p \wedge q$	$(p \wedge q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	0	1	0
1	1	1	1	1	1	1

Here columns 5 and 7 are same hence

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Hence the proof

Distributive laws:

(a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Here columns 5 and 8 are same, hence

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

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(5) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$(p \wedge (q \wedge r)$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

In the above truth table the last two columns are same, hence $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Law for the negation of a conditional:

consider the conditional $p \to q$, and its negation is obtained by using the following law:

$$\neg(p \to q) \Leftrightarrow [p \wedge (\neg q)]$$

proof: The following table gives the truth values of $\neg(p \to q)$ and $(p \wedge \neg q)$ for all possible truth values of p and q.

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

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Note: $\neg(p \to q)$ & $(p \wedge \neg q)$ have the same truth values in all possible situations.

Hence, $\neg(p \to q) \Leftrightarrow p \wedge \neg q$

Remark: As mentioned before, logically equivalent propositions are treated as identical propositions.
Accordingly, in view of the laws indicated above, we have the following results:

For any two propositions p and q.

(1) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
(2) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
(3) $\neg(p \to q) \equiv (p \wedge \neg q)$
(4) $(p \to q) \equiv \neg\neg(p \to q) \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$.

Table for negation of compound propositions:

Proposition	Negation
$\neg p$	p
$p \wedge q$	$\neg p \vee \neg q$
$p \vee q$	$\neg p \wedge \neg q$
$p \to q$	$p \wedge \neg q$

Transitive and substitution rules:

(1) If u, v, w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$ then $u \Leftrightarrow w$. → Transitive rule.

(2) Suppose that a compound proposition 'u' is a tautology and 'p' is a component of u. If we replace each occurrence of 'p' in u by a proposition q, then the resulting compound proposition 'v' is also a tautology.

→ Substitution rule

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Suppose that 'u' is a compound proposition which contains a component p. Let q be a proposition such that $q \Leftrightarrow p$. Suppose we replace one or more occurrences of p by q and obtain a compound proposition V. Then $V \Leftrightarrow u$.

$\Longrightarrow$ Substitution rule

Example (1) Let 'x' be a specified number. Write down the negation of the following conditional:

"If 'x' is an Integer, then x is rational number".

Soln: The given conditional is $p \to q$, where

p: x is an Integer. q: x is a rational number

Now, we have the result,

$$\neg(p \to q) \equiv (p \wedge (\neg q))$$

To prove, negation of the condition $p \to q$.

∴ According to the result,

$\neg(p \to q)$ is given by $(p \wedge (\neg q))$

Hence, 'x' is an Integer and x is not a rational number.

Example (2): Let x be a specified number. Write down the negation of the following proposition.

"If 'x' is not a real number, then it is not a rational number and not an irrational number."

Soln:

Consider,

p: x is a real number. q: x is a rational number,

r: x is an irrational number.

Then given proposition can be written as,

$$\neg p \to (\neg q \wedge \neg r)$$

$\therefore \neg[\neg p \to (\neg q \wedge \neg r)]$ we have $\neg(p \to q) \equiv (p \wedge \neg q)$

$\Rightarrow \neg p \wedge \neg[(\neg q \wedge \neg r)]$ we have $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

$\Rightarrow \neg p \wedge (\neg\neg q \vee \neg\neg r)$

$\Rightarrow \neg p \wedge (q \vee r)$ ∴ Negation of given proposition is:

↳ x is not a real number & it is rational or

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(3) Simplify the following compound propositions using the laws of Logic.

(i) $(p \vee q) \wedge [\neg\{(\neg p) \wedge q\}]$

$\equiv (p \vee q) \wedge \{(\neg\neg p) \vee (\neg q)\}$, By D'morgan law

$\equiv (p \vee q) \wedge \{p \vee (\neg q)\}$ By law of double negation

$\equiv p \vee \{q \wedge (\neg q)\}$ By distributive law

$\equiv p \vee F_0$, By Inverse law

$\equiv p$, By Identity law

(ii) $(p \vee q) \wedge [\neg\{(\neg p) \vee q\}]$

$\equiv (p \vee q) \wedge \{p \wedge (\neg q)\}$, By D'morgan's law

$\equiv \{(p \vee q) \wedge p\} \wedge (\neg q)$, By associative law

$\equiv \{p \wedge (p \vee q)\} \wedge (\neg q)$, By commutative law

$\equiv p \wedge (\neg q)$, By absorption law

(iii) $\neg [\neg\{(p \vee q) \wedge r\} \vee \neg q]$

$\equiv \neg [\neg\{(p \vee q) \wedge r\} \wedge q]$ By D'morgan's law

$\equiv \{(p \vee q) \wedge r\} \wedge q$, Law of double negation

$\equiv (p \vee q) \wedge (r \wedge q)$ By associative law

$\equiv (p \vee q) \wedge (q \wedge r)$ By commutative law

$\equiv \{(p \vee q) \wedge q\} \wedge r$ By associative law

$\equiv q \wedge r$, By absorption law

---

(4) prove the following logical equivalences without using truth tables:

(i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

$\Leftrightarrow p \vee p$, By absorption law

$\Leftrightarrow p$, By Idempotent law

$\equiv$ Hence the proof

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(ii) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

$[p \vee q \vee (\neg p \wedge \neg q) \wedge r]$ By associative law

$[p \vee q \vee \{\neg (p \vee q)\} \wedge r]$ By demorgan's law

$[(p \vee q) \vee \neg (p \vee q)] \wedge [(p \vee q) \vee r]$ By distribution law

$T_0 \wedge [(p \vee q) \vee r]$ By Inverse law

$T_0 \wedge (p \vee q \vee r)$ By associative law

$(p \vee q \vee r) \wedge T_0$ By commutative law

$p \vee q \vee r$ By Identity law

(iii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

WKT, $u \rightarrow v \Leftrightarrow (\neg u \vee v)$

$\therefore [(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow \neg (\neg p \vee \neg q) \vee (p \wedge q \wedge r)$

$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$, By Demorgan's law

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$, By associative law

$\Leftrightarrow p \wedge q$, By absorption law

(5) prove the following logical equivalences:

(i) $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$

consider,

$(p \vee q) \wedge (p \vee \neg q)$

$\Leftrightarrow p \vee (q \wedge \neg q)$ By distributive law

$\Leftrightarrow p \vee F_0$, as $q \wedge \neg q$ is a contradiction

$\Leftrightarrow p$, By Identity law

$\therefore [(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$

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(34) (ii) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

$\Leftrightarrow (p \to q) \wedge [\neg q \wedge (\neg q \vee r)]$ By commutative law

$\Leftrightarrow (p \to q) \wedge \neg q$, By absorption law

WKT, $\neg(u \to v) \Leftrightarrow u \wedge \neg v$

$\therefore \neg [(p \to q) \to q]$

WKT, $u \to v \Leftrightarrow \neg u \vee v$

$\therefore \neg [\neg(p \to q) \vee q]$

$\Leftrightarrow \neg [(p \wedge \neg q) \vee q]$ at $\neg(u \to v) \Leftrightarrow u \wedge \neg v$

~~$\Leftrightarrow \neg [(p \wedge \neg q) \vee q]$~~

$\Leftrightarrow \neg [q \vee (p \wedge \neg q)]$, By commutative law

$\Leftrightarrow \neg [(q \vee p) \wedge (q \vee \neg q)]$, Distribution law

$\Leftrightarrow \neg [(q \vee p) \wedge T_0]$. at $q \vee \neg q$ is a tautology

$\Leftrightarrow \neg(q \vee p)$ By Identity law.

---

(6) prove the following.

(i) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

$p \to (q \to r) \Leftrightarrow \neg p \vee (q \to r)$, at $u \to v \Leftrightarrow \neg u \vee v$

$\Leftrightarrow \neg p \vee (\neg q \vee r)$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ By associative law

$\Leftrightarrow \neg(p \wedge q) \vee r$ By Demorgan's law

$\Leftrightarrow (p \wedge q) \to r$ at $\neg u \vee v \Leftrightarrow u \to v$

(ii) $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

consider,

$[\neg p \wedge (\neg q \wedge r)]$

$\Leftrightarrow (\neg p \wedge \neg q) \wedge r$, By associative law

$\Leftrightarrow [\neg(p \vee q)] \wedge r$ Demorgan's law

$\Leftrightarrow r \wedge [\neg(p \vee q)]$ commutative law

and consider,

$(q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow (r \wedge q) \vee (r \wedge p)$ commutative law

$\Leftrightarrow r \wedge (q \vee p)$ Distributive law

$\Leftrightarrow r \wedge (p \vee q)$ commutative law

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$\therefore [\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r)]$

can be written as,

$\{r \wedge [\neg (p \vee q)]\} \vee \{r \wedge (p \vee q)\}$

$\Leftrightarrow r \wedge \{[\neg (p \vee q)] \vee (p \vee q)\}$ D

$\Leftrightarrow r \wedge T_0$, as $[\neg (p \vee q) \vee (p \vee q)$ is always true.

$\Leftrightarrow r$ By Identity law

(7) prove the following result:

$\neg [\{(p \vee q) \wedge r\} \rightarrow \neg q] \Leftrightarrow \neg [\neg (p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

we have, $\neg [\{(p \vee q) \wedge r\} \rightarrow \neg q] \Leftrightarrow \neg [\neg \{(p \vee q) \wedge r\} \wedge q]$

$\Leftrightarrow \neg \neg [\{(p \vee q) \wedge r\} \wedge q]$ as $\neg (p \vee q) \equiv (\neg p \wedge \neg q)$

$\Leftrightarrow \{(p \vee q) \wedge r\} \wedge q$, law of double negation

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ Associative law

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ commutative law

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ Associative law

$\Leftrightarrow [q \wedge (q \vee p)] \wedge r$ —"—

$\Leftrightarrow q \wedge r$ Absorption law

(8) prove that $[(p \vee q) \wedge \neg \{\neg p \wedge (\neg q \vee \neg r)\}] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

is a tautology.

Let w denote the given proposition. Then $w \equiv u \vee v$, where

$u \equiv (p \vee q) \wedge \neg \{\neg p \wedge (\neg q \vee \neg r)\}$

$v \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$u \Leftrightarrow (p \vee q) \wedge \neg \{\neg p \wedge (\neg q \vee \neg r)\}$

$\Leftrightarrow (p \vee q) \wedge \neg \{\neg p \wedge \neg (q \wedge r)\}$ as $\neg (p \wedge q) \equiv (\neg p \vee \neg q)$

$\Leftrightarrow (p \vee q) \wedge \{p \vee (q \wedge r)\}$

$\Leftrightarrow p \vee \{q \wedge (q \wedge r)\} \Leftrightarrow p \vee \{(q \wedge q) \wedge r\}$

$\Leftrightarrow p \vee (q \wedge r)$

& $v \Leftrightarrow \neg (p \vee q) \vee \neg (p \vee r)$

$\Leftrightarrow \neg \{(p \vee q) \wedge (p \vee r)\}$

$\Leftrightarrow \neg \{p \vee (q \wedge r)\}$

$\equiv \neg u$

$\therefore w = u \vee v \Leftrightarrow u \vee (\neg u)$ which is always true hence it's a tautology. //

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Duality

Suppose 'u' is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurrence of $\wedge$ & $\vee$ in u by $\vee$ & $\wedge$ respectively. Also, if u contains T_0 & F_0 as components, suppose we replace each occurrence of T_0 and F_0 by F_0 and T_0 respectively. Then the resulting compound proposition is called the dual of u and is denoted by u^d.

Eg: $u: p \wedge (q \vee \neg r) \vee (s \wedge T_0)$

Then, the dual of 'u' is given by,

$u^d: p \vee (q \wedge \neg r) \wedge (s \vee F_0)$

The following two results are of Importance:

(1) $(u^d)^d \Leftrightarrow u$ i.e, dual of dual of u is logically equivalent to 'u'.

(2) For any two propositions u and v, if $u \Leftrightarrow v$, then $u^d \Leftrightarrow v^d$ (known as principle of duality).

Eg(1): write down the duals of the following propositions:

(i) $\neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$

$u: \neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$

$u^d: \neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

(ii) $u: (p \wedge q) \vee [(\neg p \vee q) \wedge (\neg r \vee s)] \vee (r \wedge s)$

$u^d: (p \vee q) \wedge [(\neg p \wedge q) \vee (\neg r \wedge s)] \wedge (r \vee s)$

(iii) $u: (p \wedge \neg q) \vee (r \wedge T_0)$

$u^d: (p \vee \neg q) \wedge (r \vee F_0)$

(iv) $u: [(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

$u^d: [(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$

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(2) Write down the duals of the following propositions:

(i) $p \to q$

WKT, $u \to v \Leftrightarrow (\neg u \vee v)$

∴ By principle of duality,

$(p \to q) \Leftrightarrow (\neg p \vee q)$

∴ $(p \to q) \Leftrightarrow (\neg p \wedge q)$

(ii) $(p \to q) \to r$

$\Leftrightarrow \Leftrightarrow \neg(p \to q) \vee r$

$\Leftrightarrow \neg[\neg(p \vee q)] \vee r$

$\Leftrightarrow \neg(\neg p \vee q) \vee r$

$\Leftrightarrow [(p \wedge \neg q) \vee r]^d$

$\equiv (p \vee \neg q) \wedge r$

(iii) $(p \to (q \to r)$

$\Leftrightarrow [\neg p \vee (q \to r)]$

$\Leftrightarrow [\neg p \vee (\neg q \vee r)]$

∴ $[\neg p \vee (\neg q \vee r)]^q$

$\equiv [\neg p \wedge (\neg q \wedge r)]$

(3) Prove that $[(\neg p \vee q) \wedge (p \wedge (p \wedge q))] \Leftrightarrow p \wedge q$ hence deduce that $[(\neg p \wedge q) \vee (p \vee (p \vee q))] \Leftrightarrow p \vee q$

we have, $(\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$\Leftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ associative law

$\Leftrightarrow (\neg p \vee q) \wedge (p \wedge q)$ Idempotent law

$\Leftrightarrow [\neg p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)]$ Distributive law

$\Leftrightarrow [(\neg p \wedge p) \wedge q] \vee [(q \wedge q) \wedge p]$

$\Leftrightarrow (F_0 \wedge q) \vee (q \wedge p)$

$\Leftrightarrow F_0 \vee (p \wedge q)$

$\Leftrightarrow p \wedge q$

Taking the dual on both sides,

$[(\neg p \vee q) \wedge (p \wedge (p \wedge q))]^d \Leftrightarrow (p \wedge q)^d$

$\Rightarrow [(\neg p \wedge q) \vee (p \vee (p \vee q))] \Leftrightarrow (p \vee q)$

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(4) verify the principle of duality for the following logical equivalence:

$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$

Soln: The given logical equivalence is $u \Leftrightarrow v$, where

$u = \neg(p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q))$ & $v = \neg p \vee q$

$u \Leftrightarrow \neg\neg(p \wedge q) \vee (\neg p \vee (\neg p \vee q))$

$\Leftrightarrow (p \wedge q) \vee (\neg p \vee (\neg p \vee q))$

$\therefore u^d \Leftrightarrow (p \vee q) \wedge (\neg p \wedge (\neg p \wedge q))$

$\Leftrightarrow (p \vee q) \wedge (\neg p \wedge q)$

$\Leftrightarrow [p \wedge (\neg p \wedge q)] \vee [q \wedge (\neg p \wedge q)]$

$\Leftrightarrow (F_0 \wedge q) \vee (q \wedge \neg p)$

$\Leftrightarrow F_0 \vee (q \wedge \neg p)$

$\Leftrightarrow q \wedge \neg p$

also $v^d \Leftrightarrow \neg p \wedge q \Leftrightarrow q \wedge \neg p$

$\therefore u^d \Leftrightarrow v^d$

This verifies the principle of duality for the given Logical equivalence.

The connectives NAND and NOR

For any two propositions p and q, the Demorgan law state that,

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

and,

the compound proposition $\neg(p \wedge q)$ is read as "Not p and q" and is also denoted by $(p \uparrow q)$.

The symbol $\uparrow$ is called the NAND connective. Here, NAND is a combination of "Not" and "and".

The compound proposition $\neg(p \vee q)$ is read as "Not (p or q)" and is also denoted by $(p \downarrow q)$.

The symbol $\downarrow$ is called the "NOR" connective. Here NOR is a combination of Not & OR.

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Thus, $(p \uparrow q) = \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$(p \downarrow q) = \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Evidently, $(p \uparrow q)$ and $(p \downarrow q)$ are duals of each other. The combined truth table for these is shown below.

combined truth table for $(p \uparrow q)$ and $(p \downarrow q)$

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Eg: ① For any propositions p, q, prove the following:

(i) $\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$

By using the definition,

WKT, $(p \downarrow q) = \neg(p \vee q)$

$\therefore \neg(p \downarrow q) = \neg[\neg(p \vee q)]$

$\Leftrightarrow \neg[\neg p \wedge \neg q)$

$\rightarrow \Leftarrow$

$\therefore (\neg p) \uparrow (\neg q)$

(ii) $\neg(p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$

WKT, $(p \uparrow q) = \neg(p \wedge q)$

$\therefore \neg(p \uparrow q) \Leftrightarrow \neg[\neg(p \wedge q)]$

$\Leftrightarrow \neg(\neg p \vee \neg q)$

$\Leftrightarrow (\neg p \downarrow \neg q)$

② For any propositions p, q, r prove the following:

(i) $p \uparrow (q \uparrow r) \Leftrightarrow p \vee (q \wedge r)$

$p \uparrow (q \uparrow r) \Leftrightarrow \neg[p \wedge (q \uparrow r)]$

(ii) $(p \uparrow q) \uparrow r \Leftrightarrow (p \wedge q) \vee \neg r$

$\Leftrightarrow \neg[(p \uparrow q) \wedge r]$

$\neg[\neg(p \wedge q) \wedge r]$

$\neg\neg(p \wedge q) \vee \neg r$

$(p \wedge q) \vee \neg r$

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(iii) $p \downarrow (q \downarrow r) \Leftrightarrow \neg p \wedge (q \vee r)$

$\Leftrightarrow \neg [p \vee (q \downarrow r)]$

$\Leftrightarrow \neg [p \vee \neg (q \vee r)]$

$\Leftrightarrow \neg p \wedge \neg \neg (q \vee r)$

$\Leftrightarrow \neg p \wedge (q \vee r)$

(iv) $(p \downarrow q) \downarrow r \Leftrightarrow \neg [(p \downarrow q) \vee r]$

$\Leftrightarrow \neg [\{\neg (p \vee q)\} \vee r]$

$\Leftrightarrow \neg \neg (p \vee q) \wedge \neg r$

$\Leftrightarrow (p \vee q) \wedge \neg r$

Q(3). Express the following propositions in terms of only NAND and only NOR connectives.

(i) $\neg p$

Soln.: For any proposition p, we have $p \wedge p \equiv p$ & $p \vee p \equiv p$.

$\therefore \neg p \Leftrightarrow \neg (p \wedge p) \Leftrightarrow (p \uparrow p)$

also, $\neg p \Leftrightarrow \neg (p \vee p) \Leftrightarrow (p \downarrow p)$.

(ii) we have,

$p \wedge q \Leftrightarrow \neg \neg (p \wedge q) \Leftrightarrow \neg (\neg p \vee \neg q)$

$\Leftrightarrow (\neg p \vee \neg q) \uparrow (\neg p \vee \neg q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$

also,

$p \wedge q \Leftrightarrow \neg (\neg p) \wedge \neg (\neg q) \Leftrightarrow (\neg p) \downarrow (\neg q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$.

(iii) we have,

$p \vee q \Leftrightarrow \neg \neg p \vee \neg \neg q \Leftrightarrow (\neg p) \uparrow (\neg q) \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$

also

$p \vee q \Leftrightarrow \neg \neg (p \vee q) \Leftrightarrow \neg (\neg p \wedge \neg q) \Leftrightarrow \neg (p \downarrow q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$.

(iv) we have $p \rightarrow q \Leftrightarrow \neg (p \wedge \neg q) \Leftrightarrow p \uparrow (\neg q) \Leftrightarrow p \uparrow (q \uparrow q)$

also,

$p \rightarrow q \Leftrightarrow (\neg p \vee q) \Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q)$ using (iii) above

$\Leftrightarrow \{(p \downarrow p) \downarrow q\} \downarrow \{(p \downarrow p) \downarrow q\}$

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Converse, Inverse and contrapositive; Logical Implications.

Consider a conditional $p \to q$, then:

1. $q \to p$ is called the converse of $p \to q$
2. $\neg p \to \neg q$ is called the Inverse of (opposite) of $p \to q$
3. $\neg q \to \neg p$ is called contrapositive of $p \to q$.

Eg: p: 2 is an Integer. q: 9 is a multiple of 3.

Then,

$p \to q$: If 2 is an Integer, then 9 is a multiple of 3.

converse: $q \to p$: If 9 is a multiple of 3, then 2 is an Integer.

Inverse: $\neg p \to \neg q$: If 2 is not an Integer, then 9 is not a multiple of 3.

contrapositive: $\neg q \to \neg p$: If 9 is not a multiple of 3, the 2 is not an Integer.

The following table gives the truth values of

$(p \to q)$, $(q \to p)$, $(\neg p) \to (\neg q)$ and $(\neg q) \to (\neg p)$

for all possible truth values of two arbitrary propositions p and q.

Truth table for converse, Inverse, and contrapositive:

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

From this table it is evident that $p \to q$ and $(\neg q) \to (\neg p)$ have the same truth values in all possible situations.

also, $q \to p$ and $(\neg p) \to (\neg q)$ have the same truth values in all possible situations.

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As such, we have following two Important results:

(1) A conditional and its contrapositive are logically equivalent; that is, for any propositions p and q,

$$(p \to q) \Leftrightarrow (\neg q) \to (\neg p)$$

(2) The converse and the Inverse of a conditional are logically equivalent; that is, for any propositions p and q,

$$(q \to p) \Leftrightarrow (\neg p) \to (\neg q)$$

Note: $(p \to q) \not\Leftrightarrow (q \to p)$

Logical Implication:

While defining $p \to q$ no restrictions we Imposed on the choice of p and q. i.e, we can think of $p \to q$ for any two propositions p and q, as such, we may consider, for example, the propositions p,

p: 6 is a multiple of 2

& q: 3 is a prime number

i.e, $p \to q$: If 6 is a multiple of 2, the 3 is a prime number.

Note that: Here 'p' is true and 'q' is true, hence $p \to q$ is true.

But, here $p \to q$ doesn't Make any sense,

Because there is no consistency in the statement

$p \to q$ (but it is logically true).

Eg: p: 4 is an odd number and q: Bengaluru is not in Karnataka.

Both are false.

$p \to q$: If 4 is an odd number, then Bengaluru is not in Karnataka.

This conditional makes no sense but it is logically true.

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In case of logical Implication,
we do not deal with conditionals of above example, here the major Interest lies in conditional $p \to q$ where p and q are related in some way so that the truth value of 'q' depends upon the truth value of 'p' or vice-versa.

Such conditionals are called hypothetical (Implicative) statements.

* When a hypothetical statement $p \to q$ is such that q is true whenever 'p' is true, we say that p implies q.
This symbolically written as $p \Rightarrow q$, the symbol $\Rightarrow$ denoting the word implies.
* When hypothetical statement $p \to q$ is such that q is not necessarily true whenever 'p' is true, we say that p does not Imply q.
This is symbolically written as $p \not\Rightarrow q$, the symbol $\not\Rightarrow$ denoting the phrase does not Imply.

Necessary and Sufficient conditions:

Consider two propositions p and q whose truth values are Interrelated. Suppose that $p \Rightarrow q$. Then in order that q may be true it is sufficient that 'p' is true. Also if 'p' is true then it is necessary that 'q' is true. In view of this Interpretation all of the following statements are taken to carry the same meaning.

① $p \Rightarrow q$ ② p is sufficient for 'q' ③ q is necessary for 'p'

For example, the Logical Implication "If a quadrilateral is a square, then it is rectangle" is taken to have the same meaning as "(The fact that) a quadrilateral if a square is a sufficient condition for it to be a rectangle" or as "(The fact that) a quadrilateral if a rectangle is a necessary condition for it to be a square".

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For two propositions p and q, the following situations are possible:

(i) $p \Rightarrow q$, but $q \not\Rightarrow p$

(ii) $p \not\Rightarrow q$, but $q \Rightarrow p$

(iii) $p \Rightarrow q$, and $q \Rightarrow p$

* In the first case, p is sufficient but not a necessary condition for q.
* In the second case, 'p' is a necessary but not a sufficient condition for q.
* In the last case, 'p' is a necessary and sufficient condition for q and vice versa. This is expressed as 'p' if and only if q (p iff q).

Eg① 'A' denotes specified city.

consider following propositions:

p: The city 'A' is in Karnataka

q: The city 'A' is in India.

Here, $p \Rightarrow q$, but $q \not\Rightarrow p$.

i.e, 'p' is sufficient but not necessary condition for q, and 'q' is necessary but not a sufficient condition for 'p'.

Eg②: consider specified Integer 'x' and let,

p: The Integer x is even

q: The Integer 'x' is divisible by 2.

Here, $p \Rightarrow q$ and $q \Rightarrow p$. Thus here 'p' is a necessary & sufficient condition for 'q' & vice-versa. i.e, p iff q.

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(46) Eq ①: Write down the contrapositive of $[p \to (q \to r)]$ with.

(a) only one occurrence of the connective $\to$;

(b) no occurrence of the connective $\to$.

WKT, contrapositive of $[p \to (q \to r)]$ is

$$[\neg(q \to r) \to (\neg p)]$$

$$\therefore [\neg(q \to r) \to (\neg p)] \Leftrightarrow \neg\{\neg(q \to r) \vee \neg p$$

$$\Leftrightarrow (q \to r) \vee \neg p \quad — (1)$$

$$\Leftrightarrow (\neg q \vee r) \vee \neg p \quad — (2)$$

expressions (1) and (2) are the required representations.

Eq ②: prove the following:

(i) $[p \wedge (p \to q)] \Rightarrow q$ (ii) $[(p \to q) \wedge \neg q] \Rightarrow \neg p$

(iii) $[(p \vee q) \wedge \neg p] \Rightarrow q$

Soln: prepare the truth table,

p	q	$\neg p$	$\neg q$	$p \vee q$	$p \to q$
0	0	1	1	0	1
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	1	1

(1) From the table, note that when both p and $p \to q$ are true then 'q' is true. This proves that

$$[p \wedge (p \to q)] \Rightarrow q$$

(2) From the table, note that when both $p \to q$ and $\neg q$ are true, then $\neg p$ is true. This proves that

$$[(p \to q) \wedge \neg q] \Rightarrow \neg p$$

(3) From the Table, we find that when both $p \vee q$ and $\neg p$ are true, then q is true. This proves that

$$[(p \vee q) \wedge \neg p] \Rightarrow q$$

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46 (3) prove the following:

(i) $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

Soln: prepare the truth table,

p	q	r	$p \to q$	$p \vee q$	$(p \vee q) \to r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

From the table, we observe that when all of p, $p \to q$ and r are all true, then $(p \vee q) \to r$ is true. This proves that $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

(ii) $\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$

prepare the truth table,

p	q	r	$p \vee (q \vee r)$	$\neg q$	$[p \vee (q \vee r)] \wedge \neg q$	$p \vee r$
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

From rows 2, 5 and 6 of the table note that if $[p \vee (q \vee r)] \wedge \neg q$ is true the $p \vee r$ is true. This proves that $\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$ //

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Rules of Inference:

Consider a set of propositions $P_1, P_2, \cdots P_n$ and a proposition Q. Then a compound proposition of the form,

$$(P_1 \wedge P_2 \wedge P_3 \wedge \cdots P_n) \rightarrow Q$$

is called an argument. Here $P_1, P_2, \cdots P_n$ are called the premises of the argument and Q is called a conclusion of the argument.

Premises: proposition supporting the conclusion.

It is a practice to write the above argument in the following tabular form:

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \\ \hline \therefore Q \end{array}$$

Here, $(\therefore)$ → this symbol stands for "Therefore".

The preceding argument is said to be valid if whenever each of the premises $P_1, P_2, P_3 \cdots P_n$ is true, then the conclusion 'Q' is likewise true.

In other words, the argument

$$(P_1 \wedge P_2 \wedge P_3 \cdots P_n) \rightarrow Q$$

is valid when,

$$(P_1 \wedge P_2 \wedge P_3 \cdots P_n) \Rightarrow Q$$

It is ~~empha~~ to be emphasized that in an argument, the premises are always taken to be true where as the the conclusion may be true or false.

* The conclusion is true only in the case of valid argument. There exists rules of logic which can be employed for establishing the validity of arguments. These rules are called the rules of Inference.

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Some of these rules are listed below:

(1) Rule of conjunctive simplification:

This rule states that, for any two propositions p and q, if $p \wedge q$ is true, then 'p' is true. i.e,

$$(p \wedge q) \Rightarrow p.$$

This rule follows from the definition of conjunction.

(2) Rule of Disjunctive Amplification:

This rule states that, for any two propositions p and q, if 'p' is true then $p \vee q$ is true;

i.e, $p \Rightarrow p \vee q$

This rule follows from the definition of disjunction.

(3) Rule of Syllogism:

This rule states that, for any 3 propositions p, q, r, if $p \to q$ is true and $q \to r$ is true, the $p \to r$ is true.

i.e, $\{(p \to q) \wedge (q \to r)\} \Rightarrow (p \to r)$

This rule follows from the tautology $\{(p \to q) \wedge (q \to r)\} \to (p \to r)$ and is expressed in the following tabular form:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

(4) Modus ponens (Rule of detachment):
(Method of affirming)

This rule states that if 'p' is true and $p \to q$ is true, then 'q' is true.

i.e, $\{p \wedge (p \to q)\}$

tabular form,

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore \quad q \end{array} \quad //$$

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(5) Modus tollens
(method of denying)

This rule states that if $p \to q$ is true and q is false then 'p' is false i.e,

$$\{(p \to q) \wedge \neg q\} \Rightarrow (\neg p)$$

For Tabular form:

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

(6) Rule of disjunctive syllogism:

This rule states that if $p \vee q$ is true and p is false then 'q' is true. i.e,

$$\{(p \vee q) \wedge \neg q\} \Rightarrow q$$

Tabular form:

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

(7) Rule of contradiction:

This rule states that if $\neg p \to F_0$ is true, then p is true i.e, $(\neg p \to F_0) \Rightarrow p$.

Here, F_0 is a contradiction (proposition which is always false). the below truth table proves this rule.

p	F_0	$\neg p$	$\neg p \to F_0$	$(\neg p \to F_0) \to p$
0	0	1	0	1
1	0	0	1	1

This rule expressed as,

$$\begin{array}{l} \neg p \to F_0 \\ \hline \therefore p \end{array}$$

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Remarks: The validity or otherwise of a given argument may be established with the aid of the above stated rules or their appropriate combinations.

In this process, we can also employ the laws of logic, logical equivalence and/or tautologies. When there is an ambiguity or if there is no other avenue, appropriate truth tables are helpful.

Eg①: Test whether the following is a valid argument.

If Sachin hits a century, then he gets a free car.

Sachin hits a century

$\therefore$ Sachin gets a free car.

Soln.: let p: Sachin hits a century

q: Sachin gets a free car

$$\begin{array}{l} p \to q \\ \quad p \\ \hline \therefore \quad q \end{array}$$

In view of modus ponens rule, this is valid argument.

---

② Test whether the following is a valid argument:

If Sachin hits a century, he gets a free car.

Sachin does not get a free car

$\therefore$ Schin has not hit a century

Soln.. p: Sachin hits a century

q: Sachin gets a free car

then the given argument read

$$\begin{array}{l} p \to q \\ \quad \neg q \\ \hline \therefore \quad \neg p \end{array}$$

In view of modus tollens rule, the argument is valid

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(3) Test whether the following is a valid argument:

If Sachin hits a century, he gets a free car

Sachin gets a free car

∴ Sachin has hit a century

Soln. Let, p: Sachin hits a century

q: Sachin gets a free car

Then the given argument reads,

$$\begin{array}{l} p \to q \\ \quad q \\ \hline \therefore \quad p \end{array}$$

Note that if $p \to q$ and q are true, there is no rule which asserts that 'p' must be true.

Indeed p can be false when $p \to q$ and q are true.

i.e.

p	q	$p \to q$	$(p \to q) \wedge q$
0	1	1	1

Thus, $[(p \to q) \wedge q] \to p$ is not a tautology. Therefore, the given argument is not a valid one

(4) Test whether the following argument is valid:

If I drive to work, then I will arrive tired

I am not tired (when I arrive at work)

∴ I do not drive to work

let, p: I drive to work

q: I arrive tired

Then the given argument reads

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

In view of the Modus tollens rule, this is a valid argument

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(5) Test the validity of the following argument:

I will become famous or I will not become a musician

I will become a musician

∴ I will become famous

p: I will become famous

q: I will become a musician

Then, the given argument reads

$$\begin{array}{l} p \vee \neg q \\ \quad q \\ \hline \therefore \quad p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} q \to p \\ \quad q \\ \hline \therefore \quad p \end{array} \qquad (\text{as } p \vee \neg q \Leftrightarrow \neg q \vee p \Leftrightarrow q \to p)$$

In view of modus ponens rule, this argument is valid.

(6) Test whether the following is a valid argument.

If I study, then I do not fail in the examination

If I dont fail in the examination, My father gifts a two-wheeler to me

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∴ If I study then my father gifts a two-wheeler to me.

Let,

p: I study q: I do not fail in examination

r: my father gifts a two-wheeler to me

Then the given argument reads

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

In view of the Rule of Syllogism, this is a valid argument.

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Example (7)

Test the validity of the following argument:

If Ravi goes out with friends, he will not study
If Ravi doesnot study, his father becomes angry
His father is not angry

∴ Ravi has not gone out with friends

Soln. Let, p: Ravi goes out with friends
q: Ravi does not study
r: His father becomes angry

Then the given argument reads,

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \neg r \\ \hline \therefore \neg p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{ll} p \to r & \text{(using rule of syllogism)} \\ \neg r & \\ \hline \therefore \neg p & \end{array}$$

In view of modus Tollens rule, this is a valid argument.

(8) Test the validity of the following argument:

If I study, I will not fail in examination
If I do not watch TV in the evenings, I will study
I failed in the examination

∴ I must have watched TV in the evenings

Soln. Let,
p: I study , q: I fail in the examination
r: I watch TV in the evenings

i.e,
$$\begin{array}{l} p \to \neg q \\ \neg r \to p \\ q \\ \hline \therefore r \end{array}$$

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(58) This argument is logically equivalent to,

$q \to \neg p$ (because $(p \to \neg q) \iff (\neg\neg q \to \neg p)$)

$\neg p \to r$ (because $(\neg r \to p) \iff (\neg p \to r)$)

q

$\therefore r$

This is equivalent to,

$q \to r$ (using rule of syllogism)

q

$\therefore r$

This argument is valid, by the Modus ponens rule

(9) consider the following argument:

I will get grade A in this course or I will not graduate

If I do not graduate, I will join the army.

I got grade A

$\therefore$ I will not join the army

Is this valid argument?

Soln:- Let,

p: I get grade A in this course or I will not graduate

If I do not graduate, I will join the army

I got grade 'A'

$\therefore$ I will not join the army

Is this a valid argument?

Soln:- p: I get grade 'A' in this course

q: I do not graduate

r: I join the army

Then the given argument reads

$p \vee q$

$q \to r$

p

$\therefore \neg r$

This argument logically equivalent to:

$\neg q \to p$ (as $p \vee q \equiv q \vee p \iff \neg q \to p$

$\neg r \to \neg q$

p

$\therefore \neg r$

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This is logically equivalent to

$\neg r \rightarrow p$ (using rule of syllogisms)

p

$\therefore \neg r$

This is not a valid argument

(10) Test the validity of the following arguments:

(i) $p \wedge q$

$p \rightarrow (q \rightarrow r)$

$\therefore r$

WKT, the premises are always taken to be true where as conclusion may be true or false.

$\therefore$ $p \wedge q$ is true

If $p \wedge q$ is to be true, both p and q must be true.

& as 'p' is true and $p \rightarrow (q \rightarrow r)$ is $q \rightarrow r$ must be true.

'q' is true, therefore r must be true

'q' is true and $q \rightarrow r$ is true, 'r' must be true,

Hence the given argument is valid

(ii) p

$p \rightarrow \neg q$

$\neg q \rightarrow \neg r$

$\therefore \neg r$

Soln: The premises $p \rightarrow \neg q$ and $\neg q \rightarrow \neg r$ together yield the premise $p \rightarrow \neg r$. Since 'p' is true,

This premise $(p \rightarrow \neg r)$ establishes that $\neg r$ is true.

Hence the given argument is valid.

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(iii) $p \to r$
$q \to r$
$\therefore (p \vee q) \to r$

we have, $(p \to r) \wedge (q \to r)$

$\Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$
$\Leftrightarrow (r \vee \neg p) \wedge (r \vee \neg q)$ By commutative law
$\Leftrightarrow r \vee (\neg p \wedge \neg q)$ By distributive law
$\Leftrightarrow \neg(p \vee q) \vee r$, By commutative Law and Demorgan laws
$\Leftrightarrow (p \vee q) \to r$

This logical equivalence shows that given argument is valid

(11) Test whether the following arguments are valid:

(i) $p \to q$
$r \to s$
$p \vee r$
$\therefore q \vee s$

Note that, $(p \to q) \wedge (r \to s) \wedge (p \vee r)$

$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\neg p \to r)$
$\Leftrightarrow (p \to q) \wedge (\neg p \to r) \wedge (r \to s)$ commutative law
$\Leftrightarrow (p \to q) \wedge (\neg p \to s)$ Rule of syllogism
$\Leftrightarrow (\neg q \to \neg p) \wedge (\neg p \to s)$ contrapositive
$\Leftrightarrow \neg q \to s$, rule of syllogism
$\Leftrightarrow q \vee s$

This shows the given argument is valid

(ii) $(p \to q)$
$r \to s$
$\neg q \vee \neg s$
$\therefore \neg(p \wedge r)$

$(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)$
$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (q \to \neg s)$
$\Rightarrow (p \to \neg s) \wedge (r \to s)$ commutative law & rule of syllogism
$\Leftrightarrow (p \to \neg s) \wedge (\neg s \to \neg r)$, By using contrapositive
$\Rightarrow p \to \neg r$, rule of syllogism
$\Leftrightarrow \neg p \vee \neg r \Leftrightarrow \neg(p \wedge r)$

This shows that given argument is valid argument //

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(12) prove the validity of the following arguments:

(i) $p \to r$
$\neg p \to q$
$q \to s$
$\therefore \neg r \to s$

Note that $(p \to r) \wedge (\neg p \to q) \wedge (q \to s)$
$\Rightarrow (p \to r) \wedge (\neg p \to s)$ rule of syllogism
$\Leftrightarrow (\neg r \to \neg p) \wedge (\neg p \to s)$ using contrapositive
$\Rightarrow \neg r \to s$, rule of syllogism

This shows that given argument is valid

(ii) $(\neg p \vee \neg q) \to (r \wedge s)$
$r \to t$
$\neg t$
$\therefore p$

Note that $[(\neg p \vee \neg q) \to (r \wedge s)] \wedge (r \to t) \wedge (\neg t)$
$\Rightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge \neg r$, Modus tollens
$\Rightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge (\neg r \vee \neg s)$, Disjunctive amplification
$\Leftrightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge [\neg (r \wedge s)]$, D morgan law
$\Rightarrow \neg (\neg p \vee \neg q)$ Modus tollens rule
$\Leftrightarrow p \wedge q$, D morgan law
$\Rightarrow p$, rule of conjunctive simplification

This proves validity of the given argument

(13) prove that the following are valid arguments:

(i) $p \to (q \to r)$
$\neg q \to \neg p$
p
$\therefore r$

$[p \to (q \to r)] \wedge [\neg q \to \neg p] \wedge p$
$\Leftrightarrow \{[p \to (q \to r)] \wedge p\} \wedge [\neg q \to \neg p]$
$\Rightarrow (q \to r) \wedge (\neg q \to \neg p)$, Modus ponens rule
$\Leftrightarrow (q \to r) \wedge (p \to q)$, contrapositive
$\Rightarrow p \to r$, rule of syllogism
$\Rightarrow r$, Because 'p' is true (premise)

This proves the given argument is valid

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(ii) $\neg p \leftrightarrow q$

$q \rightarrow r$

$\neg r$

$\therefore p$

we find that $(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r$

$\Leftrightarrow (\neg p \leftrightarrow q) \wedge \neg q$, Modus Tollens rule

$\Leftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)] \wedge \neg q$

$\Rightarrow \{(\neg p \rightarrow q) \wedge \neg q\} \wedge (q \rightarrow \neg p)$

$\Rightarrow (\neg p \rightarrow q) \wedge \neg q$, rule of conjunctive simplification

$\Leftrightarrow \neg\neg p$, Modus Tollens

$\Leftrightarrow p$

This proves that given argument is a valid argument

(14) prove that the following are valid arguments:

(i) $p \rightarrow (q \rightarrow r)$

$p \vee \neg s$

q

$\therefore s \rightarrow r$

we have, $p \rightarrow (q \rightarrow r) \wedge (p \vee \neg s) \wedge q$

$\Leftrightarrow [p \rightarrow (q \rightarrow r) \wedge (p \vee \neg s)] \wedge q$

$\Leftrightarrow [p \rightarrow (q \rightarrow r) \wedge (s \rightarrow p)] \wedge q$ By using

$\Leftrightarrow [s \rightarrow (q \rightarrow r)] \wedge q$

$\Leftrightarrow (\neg s \wedge q) \wedge (q \rightarrow r) \wedge q$

$\Leftrightarrow (\neg s) \vee r$

$\Leftrightarrow s \rightarrow r$

(ii) $p \rightarrow (q \wedge r)$

$r \rightarrow s$

$\neg(q \wedge s)$

$\therefore \neg p$

$(r \rightarrow s) \wedge \{\neg(q \wedge s)\}$

$(r \rightarrow s) \wedge \{\neg q \vee \neg s\}$

$\Leftrightarrow r \rightarrow (\neg q)$

$\Leftrightarrow \neg r \vee \neg q \Leftrightarrow (\neg(r \wedge q))$

$\therefore \{p \rightarrow (q \wedge r)\} \wedge \{r \rightarrow s\} \wedge \{\neg(q \wedge s)\}$

$\Leftrightarrow \{p \rightarrow (q \wedge r)\} \wedge \{\neg(r \wedge q)\}$

$\Leftrightarrow \{\neg p \vee (q \wedge r)\} \wedge \{\neg(q \wedge r)\}$

$\Leftrightarrow \neg p \wedge \{\neg(q \wedge r)\}$

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(15)

(i) $(\neg p \vee q) \rightarrow r$
$r \rightarrow (s \vee t)$
$\neg s \wedge \neg u$
$\neg u \rightarrow \neg t$
$\therefore p$

$[(\neg p \vee q) \rightarrow r] \wedge [r \rightarrow (s \vee t)] \wedge (\neg s \wedge \neg u) \wedge (\neg u \rightarrow \neg t)$

$\Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge \neg s \wedge \{\neg u \wedge (\neg u \rightarrow \neg t)\}$

By rule of syllogism and associative law

$\Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge [\neg s \wedge \neg t]$, Modus ponens rule

$\Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge [\neg (s \vee t)]$, Demorgan law

$\Rightarrow \neg (\neg p \vee q)$, Modus Tollens rule

$\Rightarrow p \wedge \neg q$

$\Rightarrow p$, Rule of conjunctive simplification

This proves that the given argument is valid

(ii) $p \rightarrow r$
$r \rightarrow s$
$t \vee \neg s$
$\neg t \vee u$
$\neg u$
$\therefore \neg p$

$(p \rightarrow r) \wedge (r \rightarrow s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge (\neg u)$

$\Rightarrow (p \rightarrow s) \wedge (s \rightarrow t) \wedge (t \rightarrow u) \wedge (\neg u)$

$\Rightarrow (p \rightarrow u) \wedge (\neg u)$

$\Rightarrow \neg p$

(16) (i) $p \rightarrow q$
$q \rightarrow (r \wedge s)$
$\neg r \vee (\neg t \vee u)$
$p \wedge t$
$\therefore u$

$(p \rightarrow q) \wedge \{q \rightarrow (r \wedge s)\} \wedge \{\neg r \vee (\neg t \vee u)\} \wedge (p \wedge t)$

$\Rightarrow \{p \rightarrow (r \wedge s)\} \wedge (p \wedge t) \wedge \{(\neg r \vee \neg t) \vee u\}$

$\Rightarrow [\{p \rightarrow (r \wedge s)\} \wedge p] \wedge t \wedge [\{\neg (r \wedge t)\} \vee u]$

$\Leftrightarrow (r \wedge s) \wedge t \wedge \{\neg (r \wedge t) \vee u\}$

$\Leftrightarrow \{(r \wedge t) \wedge s\} \wedge \{\neg (r \wedge t) \vee u\}$

since left hand side is true, it follows that
r, t, s and $\neg (r \vee t) \vee u$ must be true.
since r and t are true, $\neg (r \wedge t)$ is false.
consequently, 'u' can't be false. i.e,
u has to be true.

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